A Tricky Example

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle 2y^{3/2}, 3x\sqrt{y} \rangle$ and C is the part of the parabola $y = x^2$ joining the point $P_1(-1, 1)$ to the point $P_2(2, 4)$.

We will evaluate this integral in two different ways.

Directly by parameterizing the curve

We can parameterize the indicated section of the parabola by:

$$\begin{aligned} x(t) &= t \\ y(t) &= t^2 \end{aligned} \qquad \text{for } -1 \le t \le 2$$

With these parameterizations for the x and y coordinates of points along the curve, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P(x, y)\, dx + Q(x, y)\, dy$$

where P and Q are the x and y component functions, respectively, of the field $\mathbf{F}$. Substituting in the parameterized values for x and y into these component functions and replacing dx with $x'(t)\, dt$ and dy with $y'(t)\, dt$ gives

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t=-1}^{2} 2(t^2)^{3/2}\, dt + 3t\sqrt{t^2}(2t)\, dt$$

Here's where we have to be very careful in evaluating this definite integral. Each term in the integrand involves a square root of a square $\left(\text{note that } (t^2)^{3/2} = \left(\sqrt{t^2}\right)^3\right)$.

Where t can take on positive and negative values, which it certainly does over the interval $[-1, 2]$, $\sqrt{t^2} = |t|$, not just t. Therefore, we have

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{t=-1}^{2} 2|t|^3\, dt + 6t^2\, |t|\, dt \\ &= \int_{t=-1}^{0} 2|t|^3\, dt + 6t^2\, |t|\, dt + \int_{t=0}^{2} 2|t|^3\, dt + 6t^2\, |t|\, dt \end{aligned}$$

Over the interval $[-1, 0]$, $|t| = -t$ and over the interval $[0, 2]$, $|t| = t$. Therefore, our integral becomes

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_{t=-1}^{0} -2t^3\,dt - 6t^3\,dt + \int_{t=0}^{2} 2t^3\,dt + 6t^3\,dt \\ &= \int_{t=-1}^{0} -8t^3\,dt + \int_{t=0}^{2} 8t^3\,dt \\ &= -2t^4\Big|_{-1}^{0} + 2t^4\Big|_{0}^{2} \\ &= (0-(-2)) + 2(2^4) \\ &= 34\end{aligned}$$

Using the Fundamental Theorem for Line Integrals.

The vector field $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle = \langle 2y^{3/2}, 3x\sqrt{y}\rangle$ is conservative since $P_y = 3y^{1/2} = Q_x$. Therefore, there is a scalar potential function $f(x, y)$ such that $\nabla f = \mathbf{F}$. To find such a potential f, we equate the components of ∇f with the corresponding components of $\mathbf{F}$ to get

$$\begin{aligned} f_x &= 2y^{3/2} \\ f_y &= 3xy^{1/2}\end{aligned}$$

Integrating f_x with respect to x gives $f(x, y) = 2xy^{3/2} + g(y)$, where $g(y)$ is some unknown function of y. Now differentiate this function with respect to y and compare it to the f_y shown previously. We get

$$\frac{\partial}{\partial y}\left(2xy^{3/2} + g(y)\right) = 3xy^{1/2} + g'(y)$$

But, we saw previously that $f_y = 3xy^{1/2}$, so choose $g'(y) = 0$ and so $g(y) = C$, where C is a constant. Since we need only one particular potential function, let's choose $C = 0$ as well.

Therefore, a scalar potential for the field $\mathbf{F}$ is $f(x, y) = 2xy^{3/2}$. Check for yourself that $\nabla f = \mathbf{F}$.

Finally, the Fundamental Theorem for Line Integrals says that

$$\begin{aligned}\int_C \nabla f\cdot d\mathbf{r} &= \int_{P_1}^{P_2} \nabla f\cdot d\mathbf{r} \\ &= f(P_2) - f(P_1) \\ &= f(2,4) - f(-1,1) \\ &= 2(2)(4)^{3/2} - 2(-1)(1)^{3/2} \\ &= 32 + 2 \\ &= 34\end{aligned}$$

This agrees with the result found previously by directly evaluating the line integral.