Final Exam Review Problems

1. Consider the three points $A(1, -1, 2)$, $B(1, 0, 1)$ and $C(-2, 3, 5)$.

   (a) Find parametric equations for the line that contains points A and C, then eliminate the parameter to get the symmetric equation of the line.

   (b) Find the linear equation ($ax + by + cz = d$) of the plane in 3-space which passes through the three points, then determine whether or not the point $(1, 2, 9)$ lies on the plane.

2. Let $\mathbf{a} = \langle -4, 8, 1\rangle$ and $\mathbf{b} = \langle 6, -2, 3\rangle$ be vectors in $\mathbb{R}^3$.

   (a) Find, to the nearest degree, the angle between the vectors.

   (b) Find a unit vector in the same direction as vector $\mathbf{b}$.

3. Find an equation of the plane passing through the point $(2, 4, -3)$ and having normal vector $\langle -1, 3, 2\rangle$.

4. The velocity of a particle moving in space is given by

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j} + 3\,\hat{\mathbf{k}}$$

   (a) Find the position vector $\mathbf{r}(t)$ at time $t = \pi$ if the position vector at time $t = 0$ is $\mathbf{i} + 3\,\hat{\mathbf{k}}$.

   (b) Find the distance travelled by the particle along the curve from time $t = 0$ to $t = \pi$. Hint: In a time dt the particle travels a distance of $ds = |\mathbf{v}(t)|dt$.

5. Let the vector function $\mathbf{r}(t) = (e^t \sin t)\mathbf{i} + (e^t \cos t)\mathbf{j} + (e^t)\mathbf{k}$ describe the position of a particle at time t.

   (a) Find the velocity vector and speed of the particle at time $t = 0$.

   (b) Find the distance the particle travels over the interval $0 \le t \le 3$. Hint: This is the same as the arclength of the curve $\mathbf{r}(t)$ over the same interval.

6. Verify that $u = e^{-\alpha^2 k^2 t}\sin(kx)$ is a solution to the heat conduction equation $u_t = \alpha^2 u_{xx}$, where α and k are constants.

7. The radius of a right circular cone is increasing at the rate of 1.8in/s while the height is decreasing at a rate of 2.5 in/s. Find the rate at which each of the following are changing when the radius is 120 in. and the height is 140 in.

   (a) The volume V of the cone.

   (b) The surface area S of the cone.

   Hint: For a cone, $V = \frac{1}{3}\pi r^2 h$ and $S = \pi r\sqrt{r^2 + h^2}$.

8. Consider the function $f(x, y) = -x^4 + 8x^2 - y^2 - 1$.

   (a) Find the directional derivative of f at the point $(1, 1)$ in the direction of the vector $\langle 3, -4\rangle$.

   (b) What is the maximum rate of change of f at the point $(1, 1)$ and in which direction (as given by a vector) does it occur?

   (c) Find all the critical points of f

   (d) Use the second derivative test for a function of two variables to determine the nature of f (max, min or saddle point) at each critical point.

9. The temperature T (in degrees Celcius) at point (x, y, z) is given by $T(x, y, z) = e^{-x^2-3y^2-9z^2}$, where x, y, z are measured in meters.

   (a) Find the rate of change of the temperature at the point $P(2, -1, 2)$ in the direction toward the point $(3, -3, 3)$.

   (b) In which direction does the temperature increase fastest at P?

   (c) Find the maximum rate of increase of T at P.

10. Let $f(x, y) = x^2 + 2y^2 - x^2 y$.

    (a) Find all critical points of f.

    (b) Use the second derivative test for functions of two variables to determine the nature of f at each critical point (max, min or saddle point).

    (c) Find the absolute maximum and minimum values of f on the closed disk $x^2 + y^2 \le 1$.

11. Find the Lagrange multiplier equations for the point on the surface

$$x^4 + y^4 + z^4 + xy + yz + xz = 6$$

    at which x is largest. (Do NOT solve.)

12. Give the exact value of the iterated integral below. You must do the calculations "by hand" to get credit.

$$\int_0^{\pi/6} \int_1^4 x \sin(y)\, dx\, dy.$$

13. Evaluate $\displaystyle\int_0^1 \int_0^{x^2} (x + 2y)\, dy\, dx.$

14. Evaluate the double integral $\iint_D e^{x/y}\, dA$, where D is the triangular region $\{(x, y) | x \ge 0, y \le 1, y \ge x\}$.

15. Evaluate $\displaystyle\iint_D e^{x^2+y^2}\, dA$, where D is the region within the circle $x^2 + y^2 = 1$ above the x-axis. Hint: Use polar coordinates.

16. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \mathbf{i} + ye^{x/y}\mathbf{j}$ and C is the positively oriented triangle with vertices $(0, 0)$, $(1, 1)$ and $(0, 1)$.

17. Evaluate $\displaystyle\int_C xy^3\, ds$, where C is the curve parameterized by $x = 4\sin t$, $y = 4\cos t$, $z = 3t$, for $0 \le t \le \frac{\pi}{2}$.

18. Let $\mathbf{F}(x, y) = \left\langle 2xy, x^2 + y \right\rangle$ be a vector field and let C be a curve given by $\mathbf{r}(t) = \left\langle t^2, t/\sqrt{t+2} \right\rangle$, for $0 \le t \le 2$.

    (a) Show that $\mathbf{F}$ is a conservative field.

    (b) Find a potential function f, such that $\nabla f = \mathbf{F}$.

(c) Evaluate the line integral below either directly, or by replacing the given path C with a simpler path connecting the same endpoints or by using the potential function found above.

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

19. Let $\mathbf{F}(x, y) = \langle xy, x \rangle$ be a (non-conservative) vector field and let C be the unit circle described by $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, for $0 \leq t \leq 2\pi$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ by:

    (a) doing the actual line integral.

    (b) Using Green's Theorem.

20. Let $\mathbf{F}(x, y) = \left\langle 3xy^2 + 4x, 3x^2y - 4y \right\rangle$ be a vector field representing the flow of a fluid in the plane. Let C be the positively oriented boundary of the semi-circular disk bounded by the x-axis and the circular arc $y = \sqrt{4 - x^2}$.

    (a) Find the net circulation of the field F around the curve C.

    (b) Find the net flux of the field F through the curve C. Hint: Use the normal form of Green's Theorem: Flux of F through $C = \oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \iint_D \operatorname{div}\mathbf{F}\, dA$